ON THE DIAGNOSABILITY OF SYSTEMS WITH SELF-TESTING UNITS

Hideo Fujitwara and Hiroshi Ozaki

Department of Electronic Engineering
Osaka University
Yamada-Kami, Suita-Shi
Osaka, 565 JAPAN

Abstract

The diagnostic model introduced by Preparata et al. assumes that 1) each unit has the capability of testing any other unit; and 2) no unit tests itself. In this paper we eliminate these assumptions and consider a new diagnostic model in which there exist some units without capability of testing other units and some units with capability of testing themselves. For this model we consider the following three problems: 1) Necessary and sufficient conditions for the existence of testing links in a self-diagnosable systems. 2) Methods for optimal assignment of testing links to achieve a given diagnosability. 3) Necessary and sufficient conditions for a given system to be self-diagnosable.

I. Introduction

While the investigation of self-diagnosable computer architectures has been proceeded [1]-[3], many studies have been concerned with the analysis and synthesis problems of self-diagnosable systems in graph-theoretic models [4]-[11]. The graph-theoretic model was first formulated by Preparata et al. [4], and it assumes that 1) each unit has the capability of testing any other unit; and 2) no unit tests itself.

However, in the real computer systems these assumptions may not always hold. Hence in order to have a more realistic assumption, we eliminate the above assumptions and consider a new diagnostic model in which there exist some units without capability of testing other units and some units with capability of testing themselves. If a test outcome of a faulty unit is unreliable, then self-testing may or may not yield any information. Hence, we assume that the outcome of self-testing is always reliable; that is if the test outcome of a self-testing unit is good, then we can conclude that the unit is fault-free. For this model we consider the following three problems: 1) Necessary and sufficient conditions for the existence of testing links to form t-diagnosable systems. 2) Methods for optimal assignment of testing links to achieve a given diagnosability. 3) Necessary and sufficient conditions for a given system to be t-diagnosable.

II. Diagnostic Model

A system is supposed to be partitioned into n subsystems, or units, not necessarily identical. Some units have the capability of testing other units of the system by applying stimuli and observing the ensuing responses. Some units have the capability of testing themselves. Some units do not have the capability of testing. Let \( V = \{ u_1, u_2, \ldots, u_n \} \) be the set of all units. Let \( V_s \subseteq V \) be the set of units without capability of testing. Let \( V_t \subseteq V \setminus V_s \) be the set of units that are capable of testing themselves and other units. Each unit in \( V_t \) is capable of testing other units but not itself. The relation of testability can be represented by a function \( \Gamma \) mapping \( V \) into \( 2^V \) such that \( u \subseteq \Gamma(u) \).

If and only if \( u \) \( \Gamma(u) \). Hence, a system \( S \) will be represented by a quadruple \( S = (V, V_t, V_s, \Gamma) \). When \( \Gamma \) is undefined, it is denoted by the dash, that is, \( S = (V, V_t, V_s, \Gamma) \). Clearly, this diagnostic model can be also represented by a directed graph \( \phi(V, \Gamma) \), called a diagnostic graph, where the arc \((u, u_\Gamma)\) is an ordered pair of vertices in \( V \) such that \( u_\Gamma = \Gamma(u) \). For \( X \subseteq V \), we define the following functions: \( \Gamma(X) = \bigcup_{u \in X} \Gamma(u) \), \( \Delta(X) = \Gamma(X) \setminus X \), \( \Gamma^{-1}(X) = \{ u \mid \Gamma(u) \subseteq X \} \), \( \Delta^{-1}(X) = \Gamma^{-1}(X) \setminus X \).

Each unit in \( V_t \) can test itself and other units, and each unit not in \( V_t \) cannot test itself. Therefore, the function \( \Gamma \) must satisfy the following connection condition.

**Connection Condition:**
1. For \( u, \Gamma(u) = \emptyset \) (the empty set).
2. For \( u, \Gamma(u) = \emptyset \).

The testing unit \( u_\Gamma \) evaluates the tested unit \( u \) as either fault-free or faulty. The test outcome is indicated by the weight \( \omega(u, u_\Gamma) \) of the arc \((u, u_\Gamma)\) of \( G \) and obeys the following rule:

For \( u, u_\Gamma \):

\[ \omega(u, u_\Gamma) = 0 \quad \text{if} \quad u_\Gamma \text{ and } u, \text{ are fault-free,} \]

\[ \omega(u, u_\Gamma) = 1 \quad \text{if} \quad u_\Gamma \text{ is fault-free and} \]

\[ \omega(u, u_\Gamma) = 1 \quad \text{if} \quad u \text{ is faulty,} \]

\[ \omega(u, u_\Gamma) = 0 \quad \text{if} \quad u \text{ is faulty and } u_\Gamma \text{ is}

\[ \omega(u, u_\Gamma) = 0 \quad \text{if} \quad u \text{ and } u_\Gamma \text{ are faulty.} \]

For \( u_\Gamma, u_\Gamma \):

\[ \omega(u_\Gamma, u_\Gamma) = 0 \quad \text{if} \quad u_\Gamma \text{ and } u_\Gamma \text{ are fault-free,} \]

\[ \omega(u_\Gamma, u_\Gamma) = 1 \quad \text{if} \quad u_\Gamma \text{ is fault-free and} \]

\[ \omega(u_\Gamma, u_\Gamma) = 0 \quad \text{if} \quad u_\Gamma \text{ is faulty.} \]
This weight function \( \sigma \) is termed syndrome of the system. Given a directed graph \( G=(V,F) \) of a system \( S \) and a syndrome \( \sigma \) of \( S \), the fundamental problem is to identify the faulty units. A consistent fault set \( F \) with respect to \( \sigma \) must satisfy the following conditions:

1. \( \sigma(u_j,u_k) = 1 \) for all \( u_j, u_k \) such that \( u_j \in F \) and \( u_k \notin F \).
2. \( \sigma(u_j,u_k) = 0 \) for all \( u_j, u_k \) such that \( u_j \in F \) and \( u_k \in F \).
3. \( \sigma(u_j,u_k) = 1 \) for all \( u_j \) such that \( u_j \notin F \) and \( u_k \in F \).

A system \( S=(V,V_1,V_2,T) \) is said to be one-step t-fault diagnosable (shortly, t-diagnosable) if for any syndrome \( \sigma \), and at most one consistent fault set \( F \) with respect to \( \sigma \) such that \( |F| \leq t \), where \( |X| \) denotes the cardinality of a set \( X \), all faults in \( F \) can be located (identified).

### III. Existence Condition for Diagnosable Systems

In this section, we consider the following problem: Given a system \( S=(V,V_1,V_2,T) \); find necessary and sufficient condition for the existence of a function \( \Gamma \) to form a t-diagnosable system. Note that the function \( \Gamma \) must satisfy the connection condition as mentioned above.

**Theorem 1:** Given a system \( S=(V,V_1,V_2,T) \):

1. in case of \( |V_2| \geq t \), there always exists a function \( \Gamma \) such that \( S=(V,V_1,V_2,T) \) is t-diagnosable, and
2. in case of \( |V_2| < t \), there exists a function \( \Gamma \) such that \( S=(V,V_1,V_2,T) \) is t-diagnosable if and only if \( |V_1| - |V_2| - |V_1|-1 \geq 2t+1 \).

**Proof:** The proof in case of \( |V_2| \geq t \) is obvious.

In case of \( |V_2| < t \), the sufficiency can be proved as follows. Let \( \Gamma \) be a function such that \( \Gamma^{-1}(u) = \{u\} \) for \( u \in V_2 \) and \( \Gamma^{-1}(u) = V \setminus \{u\} \) for \( u \in V \setminus V_2 \).

Clearly, \( \Gamma \) satisfies the connection condition.

If there exists at least one fault-free unit in \( V_2 \), we can uniquely find a consistent fault set. If all the units in \( V_2 \) are faulty, then we can see that there exist at most \( t - |V_2| \) faulty units in \( V \setminus V_2 \). Since \( |V_1| - |V_2| - |V_1|-1 \geq 2t+1 \), the subsystem consisting of \( V, V_2, V_1 \) is \( t - |V_2| \)-diagnosable. Therefore all the faulty units in \( V \setminus V_1, V_2 \) can be identified. On the other hand, \( t > |V_2| \) implies \( |V_1| - |V_2| - |V_1|-1 \geq t \), and thus we can find at least one fault-free unit \( u \) in \( V \setminus V_1 \). Using this unit \( u \), each unit in \( V_2 \) can be evaluated as either fault-free or faulty. In this way we can uniquely find a consistent fault set. Hence, the system \( S=(V,V_2,V_1,T) \) is t-diagnosable.

Conversely, the necessity can be proved as follows. Consider a maximally connected graph \( G=(V,F) \) satisfying the connection condition, that is, \( \Gamma^{-1}(u) = V \setminus V_1 \) for all \( u \in V_2 \), \( \Gamma^{-1}(u) = V \setminus V_2 \) for all \( u \in V \setminus V_1 \), and \( \Gamma^{-1}(u) = u \) for all \( u \in V \). We shall prove that if \( |V_1| - |V_2| - |V_1|-1 \leq 2t+1 \), then \( \hat{S}=(V,V_1,V_2,T) \) is t-diagnosable. Suppose a syndrome \( \sigma \) such that \( \sigma(u,v)=1 \) for all \( u \in V \setminus V_2 \) and \( \sigma(u,v)=0 \) for all \( u \in V_2 \), \( \forall v \). Then it can easily be seen that all units in \( V_2 \) are faulty and all units in \( V_1 \) are fault-free. This implies that there exist at most \( |V_2| \) faulty units in \( V \setminus V_1 \).

Since \( |V_1| - |V_2| - |V_1|-1 < t+1 \), the subsystem consisting of \( V \setminus V_2 \) is \( t - |V_2| \)-diagnosable. Hence we cannot identify all the faulty units in \( V \setminus V_1 \). This implies that \( S \) is not t-diagnosable. This is a contradiction.

**Corollary 1:** Given a system \( S=(V,V_1,V_2,T) \), then there exists a function \( \Gamma \) such that \( S=(V,V_1,V_2,T) \) is t-diagnosable if and only if \( t \leq \min \{|V_1|, \frac{|V_1| - |V_2| - |V_1|-1}{2} \} \).

When both \( V_1 \) and \( V_2 \) are empty, we can show that the condition of Theorem 1 and Corollary 1 means \( |V| = 2t+1 \). This coincides with Theorem 1 of Preparata et al. [4], which is a special case of Theorem 1 and Corollary 1.

### IV. Optimal Connection Assignments

In the last section, we have shown the necessary and sufficient condition for the existence of a function \( \Gamma \) to form a t-diagnosable system. In this section we consider the design of optimal t-diagnosable systems provided that the above condition holds. The optimal t-diagnosable system is defined as a t-diagnosable system in which the number of arcs is minimum.

In the model of Preparata et al. [4], if a system \( S \) is t-diagnosable, then each unit of \( S \) is tested by at least \( t \) other units. In our model we have the following results as follows. Let \( S=(V_1,V_2,T) \) be a t-diagnosable system. Let \( U \subseteq V_2 \) be the set of all self-testing units, that is, \( U = \{u | u \in F(u)\} \). Then we have the following lemma.

**Lemma 1:** If \( S \) is t-diagnosable, then \( \Gamma^{-1}(u) \geq t \) for all \( u \in U \).

From Lemma 1, we can see that the number of arcs of a t-diagnosable system is at least \( (|V_1| - |U|) \times |U| \). Since \( U \subseteq V_2 \), we have \( (|V_1| - |U|) t + |U| \geq (|V_1| - |V_2|) t + |V_2| \).

Therefore, we have the following lemma.

**Lemma 2:** If \( S=(V,V_1,V_2,T) \) is an optimal t-diagnosable system, then the number of arcs of \( S \) is at least \( (|V_1| - |V_2|) t + |V_2| \).
When \( V_2 \) is empty, Lemma 2 coincides with the result of Preparata et al. [4].

Theorem 2: If \( S(V_1, V_2, V_3, \Gamma) \) is t-diagonizable, then \( S(V_1, V_2, V_3, \Gamma) \) is also t-diagonizable, where \( \Gamma \) is defined as follows:

- \( \Gamma^{-1}(u) = \{u \} \) for all \( u \in V_2 \),
- \( \Gamma^{-1}(u) = \{u \} \) for all \( u \in V_2 \).

Proof: Assume that \( S \) is not t-diagonizable. Then, there exist two consistent fault sets \( F_1 \) and \( F_2 \) with respect to a syndrome \( \sigma \) such that 

- \( \sigma \cdot F_1 \geq t \) and \( \sigma \cdot F_2 \geq t \).

From \( \sigma \cdot F_1 \) we define a syndrome \( \sigma \) for the system \( S \) as follows:

- For \( u_j \in \sigma(u_1) \), \( u_j \in V_1 \) and \( u_j \in \sigma(u_2) \), \( \sigma(u_j) = \sigma(u_1) \).
- For \( u_j \in \sigma(u_1) \), \( u_j \in V_1 \) and \( u_j \in \sigma(u_2) \), \( \sigma(u_j) = \sigma(u_1) \).

Then, we can see that both \( F_1 \) and \( F_2 \) are also consistent fault sets of \( S \) with respect to the syndrome \( \sigma \). Therefore, \( S \) is not t-diagonizable.

Q.E.D.

Now we consider the following problem: Given a system \( S = (V_1, V_2, V_3, \Gamma) \), find methods for optimal assignment of testing links \( \Gamma \) so that \( S(V_1, V_2, V_3, \Gamma) \) is t-diagonizable. On the basis of Theorem 2, we can design a class of optimal t-diagonizable systems using the optimal design \( D_{\text{opt}} \) introduced by Preparata et al. [4] as follows. A system \( S \) is said to belong to a design \( D_{\text{opt}} \) when a testing link from \( u_i \) to \( u_j \) exists if and only if \( 1-|\sigma(u_i)\sigma(u_j)| \equiv 0 \pmod{m} \) and \( m \) assumes the values 1, 2, ..., \( t \).

Method of Design \( D_{\text{opt}} \):

1. For the set \( V_1 \in V_2 \), construct a graph \( G_0 = (G_1, G_2) \) such that the system corresponding to the graph \( G_0 \) is one of the optimal system \( D_{\text{opt}} \) of Preparata et al. [4].
2. Using the graph \( G_0 \), construct a graph \( G(V_2, \Gamma) \) such that

- \( \Gamma^{-1}(u) = \{u\} \) for all \( u \in V_2 \),
- \( \Gamma^{-1}(u) = \{u\} \) for all \( u \in V_2 \),
- \( \Gamma^{-1}(u) = \{u\} \) for all \( u \in V_2 \),
- \( \Gamma^{-1}(u) = \{u\} \) for all \( u \in V_2 \).

A system constructed by the above method is said to belong to a design \( D_{\text{opt}} \). From Theorem 2, it can easily be seen that a system \( S(V_1, V_2, V_3, \Gamma) \) belonging to design \( D_{\text{opt}} \) is t-diagonizable. Moreover, the number of the system \( S \) is in exactly \( |V_1| \times \Gamma \cdot |V_2| \). Therefore from Lemma 2 we can see that the system \( S \) is optimal.

Next we shall show another optimal design of t-diagonizable systems.

Method of Design \( D_{\text{opt}} \):

(Case of \( |V_2| \geq t \))

Construct a graph \( G(V_1, \Gamma) \) such that

- \( \Gamma^{-1}(u) = \{u\} \) for all \( u \in V_2 \),
- \( \Gamma^{-1}(u) = \{u\} \) for all \( u \in V_2 \),
- \( \Gamma^{-1}(u) = \{u\} \) for all \( u \in V_2 \),
- \( \Gamma^{-1}(u) = \{u\} \) for all \( u \in V_2 \).

A system constructed by the above method is said to belong to a design \( D_{\text{opt}} \). It can be proved similarly as the proof of Theorem 1 that a system \( S \) belongs to the design \( D_{\text{opt}} \) is t-diagonizable. Moreover the number of arcs of the system \( S \) is exactly \( |V_1| \times \Gamma \cdot |V_2| \). Therefore from Lemma 2 we can see that the system \( S \) is optimal. When both \( V_1 \) and \( V_2 \) are empty, the designs \( D_{\text{opt}} \) and \( D_{\text{opt}} \) coincide with the design \( D_{\text{opt}} \) of Preparata et al. [4].

Example: To illustrate the design of \( D_{\text{opt}} \) and \( D_{\text{opt}} \), consider a system \( S(V_1, V_2, V_3, \Gamma) \) where \( V_1 = \{u_1, u_2, \ldots, u_m\} \), \( V_2 = \{v_1, v_2\} \), \( V_3 = \{v_1, v_2\} \). Suppose that \( t = 2 \), then we have a system which belongs to \( D_{\text{opt}} \) and \( D_{\text{opt}} \) as shown in Fig. 1. A system belonging to \( D_{\text{opt}} \) is shown in Fig. 2.

V. Diagonizable Systems

So far we have discussed optimal connection assignment problem for t-diagonizable systems. In this section we present the necessary and sufficient condition for a system with a diagnostically graph to be t-diagonizable. For the model of Preparata et al. [4], the necessary and sufficient condition for t-diagonizable was already shown by Allan et al. [6]. In this section we extend their results to our new model.

Given a system \( S(V_1, V_2, V_3, \Gamma) \) and its graph \( G(V_1, \Gamma) \), let \( U \subseteq V_2 \) be the set of all self-testing units, that is, \( U = \{u \in \cup \Gamma(u)\} \). Let \( P(G) \) be the set of all partitions of \( V \) with three disjoint blocks \( (X, Y, Z) \), such that

\[ |X| \leq 1, \]

\[ \Delta(X) \subseteq X \quad \text{i.e.}, \quad \Delta(X) \subseteq Y \cup Y, \]

\[ U \subseteq X \cup Y. \]

Let \( \kappa \) be a function from \( P(G) \) to the set of all positive integers such that for any partition \( (X, Y, Z) \)

\[ \kappa(p) = |X| + |Z|/2. \]

For a graph \( G(V_1, \Gamma) \), we define

\[ \kappa(p) = \min \{ k(p) \mid p \in P(G) \} - 1, \]

but \( \kappa(p) = |V| \) when \( P(G) \) is empty.

Theorem 3: Given a system \( S(V_1, V_2, V_3, \Gamma) \) and its graph \( G(V_1, \Gamma) \), then the system \( S \) is t-diagonizable if and only if \( \kappa(p) > t \) for all \( p \in P(G) \), that is, \( k(p) \leq t \).
Proof—Necessity: Assume that there exists a partition \( p = (X_1, X_2) \) of \( P(G) \) with \( k(p) \leq t \). Divide the set \( X_2 \) into two disjoint sets \( X_1 \) and \( X_2' \) such that \( |X_1'| \leq \left\lfloor \frac{|X_2'|}{2} \right\rfloor \) and \( |X_2'| \leq \left\lfloor \frac{|X_2'|}{2} \right\rfloor \). Let \( X_1' \cup X_2' \), and \( X_2' \cup X_2' \), then we have \( |X_1'| = |X_2'| \leq k(p) \leq t \) and \( |X_2'| = |X_2'| \leq k(p) \leq t \).

For these sets \( X_1' \) and \( X_2' \), we define a syndrome \( s \) as follows:

- \( s(u,v) = 1 \) if \( u \in X_1' \) and \( v \in X_2' \)
- \( s(u,v) = 1 \) if \( u \in X_2' \) and \( v \in X_1' \)
- \( s(u,v) = 0 \) otherwise.

Then we can easily see that both \( X_1' \) and \( X_2' \) are consistent fault sets with respect to the syndrome \( s \). Therefore, the system \( S \) is not \( t \)-diagnosable.

Sufficiency: Assume that \( S \) is not \( t \)-diagnosable. Then there exist two consistent fault sets \( X_1' \) and \( X_2' \) with respect to a syndrome \( s \) such that \( X_1' \neq X_2' \), and \( |X_1'| \leq t \) and \( |X_2'| \leq t \). Let \( Y \equiv (X_1', X_2') \), \( Y \equiv X_1' \cup X_2' \), and \( Z \equiv (X_1', X_2') \). Since \( \phi \neq \phi \), we have \( \phi \neq \phi \).

Since both \( X_1' \) and \( X_2' \) are consistent fault sets, we have \( \lambda(X_1', Y) \leq X \) and \( \lambda(Y, X_1') \leq X \). Therefore, \( \lambda = (X_1', X_2') \) is in \( P(G) \). Moreover, \( 2|Y| - |X_1'| - |X_2'| \leq 2t \).

Hence we have \( k(p) = \left| X_1' \cup X_2' \right| / 2 \leq t \). Q.E.D.

When a system is \( t \)-diagnosable but not \((t+1)\)-diagnosable, then the number \( t \) is called the diagnosis of the system. From Theorem 3, we can see that \( \tau(G) \) represents the diagnosability of the diagnostic graph \( G \). When the set \( U \) is empty, the system \( P(G) \) coincides with the set of partitions introduced by Allan et al. [6]. Therefore, Theorem 3 is a generalization of the result of Allan et al. When \( V \equiv \emptyset \), that is, all the units are self-testing units, then \( P(G) \) is empty, and thus the condition of Theorem 3 always holds and \( S \) is \( t \)-diagnosable for any \( t \leq |V| \).

For the diagnosability \( \tau(G) \) we have the following theorem.

**Theorem 4:** Given a system \( S \equiv (V, V_1, V_2), T \) and its graph \( G = \langle V, E \rangle \), then

\[
\tau(G) \leq \left| \frac{|V| + |V_2| - |V_1| - 1}{2} \right|
\]

otherwise.

**Proof:**

**Case 1:** \( V = V_1 \cup V_2 \).

When \( V_1 \) is not empty, let \( p_0 = (V_1, \gamma(V_1, V_2)) \) for some \( \gamma \in E \). Then it can easily be seen that \( p_0 \in P(G) \) and that

\[
\min \{ k(p) \mid p \in P(G) \} \leq k(p_0) = |V_2| + \left| \frac{1}{2} - |V_2| + 1 \right|
\]

Therefore we have \( \tau(G) \leq |V_2| \).

When \( V_1 \) is empty, then \( \gamma \in E \) and thus we have

\[
\tau(G) \leq |V_2|.
\]

**Case 2:** \( V \neq V_1 \cup V_2 \).

Let \( p_0 = (V_1, V_2, \gamma(V, \gamma(V_1, V_2))) \), then it can easily be seen that \( p_0 \in P(G) \). Moreover we have

\[
\min \{ k(p) \mid p \in P(G) \} \leq k(p_0) = |V_2| + \left| \frac{1}{2} - |V_2| + 1 \right|
\]

This implies that

\[
\tau(G) \leq |V_2| + \left| \frac{1}{2} - |V_2| + 1 \right|
\]

Q.E.D.

**Corollary 2:** Given a system \( S \equiv (V, V_1, V_2, T) \) and its graph \( G = \langle V, E \rangle \), then

\[
\tau(G) \leq \max \{ \left| V_1 \right|, \left| V_2 \right|, \left| V_2 \right| - \left| V_1 \right| \}
\]

Q.E.D.

**Corollary 3:** Given a system \( S \equiv (V, V_1, V_2, T) \) and its graph \( G = \langle V, E \rangle \), then

\[
\tau(G) \leq \max \{ \left| V_1 \right|, \left| V_2 \right|, \left| V_2 \right| - \left| V_1 \right| \}
\]

Q.E.D.

**Corollary 4:** Given a system \( S \equiv (V, V_1, V_2, T) \) and its graph \( G = \langle V, E \rangle \), then

\[
\tau(G) \leq \max \{ \left| V_1 \right|, \left| V_2 \right|, \left| V_2 \right| - \left| V_1 \right| \}
\]

Q.E.D.

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**References**


![Fig. 1 Design D_{12}](image1)

![Fig. 2 Design D_{12}](image2)